FUNCTIONS AND INVERSE FUNCTIONS

A FUNCTION is a relationship or a rule between the input (x-values/domain) and the output (y-values/range)

<table>
<thead>
<tr>
<th>Input-value</th>
<th>output-value</th>
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<tbody>
<tr>
<td>2</td>
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<td>0</td>
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<td>-2</td>
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The INVERSE FUNCTION is a rule that reverses the input and output values of a function.

If \( f \) represents a function, then \( f^{-1} \) is the inverse function.

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Functions can be \textbf{on – to – one} or \textbf{many – to – one} relations.

\textbf{NOTE:} if a relation is \textbf{one – to – many}, then it is \textbf{NOT} a function.
HOW TO DETERMINE WHETHER THE GRAPH IS A FUNCTION OR NOT

i. **Vertical – line test:**

The **vertical – line test** is used to determine whether a graph is a function or not a function. To determine whether a graph is a function, draw a vertical line parallel to the y-axis or perpendicular to the x-axis. If the line intersects the graph once then graph is a function. If the line intersects the graph more than once then the relation is not a function of x. Because functions are single-valued relations and a particular x-value is mapped onto one and only one y-value.

![Vertical line test](image1)

**Function**

![Vertical line test](image2)

**not a function (one to many relation)**

**TEST FOR ONE –TO– ONE FUNCTION**

ii. **Horizontal – line test**

The **horizontal – line test** is used to determine whether a function is a one-to-one function. To determine whether a graph is one –to– one function, draw a horizontal line parallel to the x-axis or perpendicular to the y-axis. If the line intersects the graph once the graph is one –to– one function. If the line intersects the graph more than once then the relation is **not a** one –to– one function.

![Horizontal line test](image3)

**one-to-one function**

![Horizontal line test](image4)

**many-to-one function**
MATHEMATICS GRADE 12

INVERSE FUNCTIONS

ACTIVITY 1 (50 marks)

1. State whether the following relations are functions or not. If the graph is a function, state whether the function is one-to-one or many-to-one.

1.1 \{(-2; -7); (0; -1); (1; 2); (2; 6)\}

1.2 \{(-2; 6); (-1; 3); (0; 2); (1; 3); (2; 6)\}

1.3 \{(-2; 16); (4; 1); (4; 6); (3; 7)\}

1.4

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>-2</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>2</td>
</tr>
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<td>2</td>
<td>5</td>
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</table>

1.5

1.6

[Graphs and tables as shown in the document]
2. Sketch the following functions:

2.1 \( f(x) = -x + 1 \)  
2.2 \( g(x) = -x^2 + 4 \)  
2.3 \( h(x) = \left( \frac{1}{3} \right)^x - 1 \)  
2.4 \( k(x) = \sqrt{x} + 1 \)  
2.5 \( j(x) = \frac{2}{x+1} + 2 \)

3. For the graphs sketched in question 2 above, state the domain and Range

4. For each of the functions given in question 2 above, write the equation of the new function formed after a translation of 1 unit right and 2 units down.

5. Explain why \( h(x) \) is a function and state with a reason(s) why it is a 1-to-1 function.

TOTAL: 50

METHOD ON HOW TO DETERMINE THE EQUATION OF THE INVERSE

- First interchange/swap \( x \) and \( y \),
- then make \( y \) the subject of the formula

**Example1: Linear function**

Determine the inverse of \( f(x) = 2x + 3 \)

**Solution**

\[ y = 2x + 3 \]

\[ x = 2y + 3 \] Interchange \( x \) and \( y \) this is also the inverse but is in the form \( x = \ldots \)

\[ x - 3 = 2y \]

\[ \frac{x-3}{2} = y \] this is in the form \( f^{-1}(x) = \frac{x-3}{2} \) or \( y = \ldots \)
Sketch the graphs of \( f(x) = 2x + 3 \) and \( f^{-1}(x) = \frac{x-3}{2} \) on the same set of axes.

Both \( f \) and \( f^{-1} \) intersect at a point \((-3; -3)\). The line \( y = x \) is the axis of symmetry.

<table>
<thead>
<tr>
<th>Domain</th>
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<tr>
<td>( f(x) )</td>
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</tr>
<tr>
<td>( f^{-1}(x) )</td>
<td>( x \in \mathbb{R} )</td>
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- Both \( f \) and \( f^{-1} \) have the same domain and range but the \( y \)-intercept of \( f \) is now the \( x \)-intercept of \( f^{-1} \).
- \( f(x) \) and \( f^{-1}(x) \) are both one-to-one functions.

Example 2: Quadratic function

Determine the inverse of \( f(x) = 2x^2 \)

Solution:
\[
\begin{align*}
y &= 2x^2 \\
x &= 2y^2 & \text{Interchange } x \text{ and } y \text{ this is also the inverse but is in the form } x = \cdots \\
\frac{x}{2} &= y^2 \\
\pm \sqrt{\frac{x}{2}} &= y \\
\pm \sqrt{\frac{x}{2}} &= y & \text{this is in the form } f^{-1}(x) = \pm \sqrt{\frac{x}{2}} \text{ or } y = \cdots
\end{align*}
\]
Sketch the graphs of \( f(x) = 2x^2 \) and \( f^{-1}(x) = \pm \frac{x}{2} \) on the same set of axes.

Both \( f \) and \( f^{-1} \) intersect at two points. The line \( y = x \) is the axis of symmetry.

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- From the sketch above, the domain and range of \( f(x) \) have interchanged forming range and domain respectively of \( f^{-1}(x) \)
- But \( f^{-1}(x) \), the inverse of \( f(x) \), is NOT a function because according to the vertical line test the graph of \( f^{-1}(x) \) is cut twice by the vertical line.
- \( f^{-1}(x) \) is a one-to-many relation.

If the domain of \( f(x) \) is restricted to \( x \geq 0 \) or \( x \leq 0 \), then the inverse will also be a function.

**Restriction 1**

For \( f(x) \)

**Domain:** \( x \geq 0 \)  \( y \geq 0 \)

For \( f^{-1}(x) \)

**Domain:** \( x \geq 0 \)  \( y \geq 0 \)
Example 3: Exponential function

Determine the inverse of \( f(x) = 2^x \)

Solution:

\[ y = 2^x \]
\[ x = 2^y \]

Interchange \( x \) and \( y \) this is also the inverse but is in the form \( x = \ldots \)

\[ \log_2 x = \log_2 2^y \]

introduce logarithm to the base of 2 on both sides of the equation

\[ \log_2 x = y \log_2 2 \]

but \( \log_a a = 1 \) \( \Rightarrow \log_2 2 = 1 \)

\[ \log_2 x = y \]

this is in the form \( f^{-1}(x) = \log_2 x \) or \( y = \ldots \)

Sketch graphs of \( f(x) = 2^x \) and \( f^{-1}(x) = \log_2 x \) on the same set of axes.
Both $f$ and $f^{-1}$ are one-to-one functions. The line $y = x$ is the axis of symmetry.

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**Example 4: Exponential function**

Determine the inverse of $f(x) = \left(\frac{1}{2}\right)^x$

**Solution:**

$y = \left(\frac{1}{2}\right)^x \quad x = 2^y$ Interchange $x$ and $y$ this is also the inverse but is in the form $x = \ldots$

$\log_2 x = \log_2 \left(\frac{1}{2}\right)^y$ introduce logarithm to the base of $\frac{1}{2}$ on both sides of the equation

$\log_2 x = y \log_2 \frac{1}{2}$ but $\log_a a = 1 \Rightarrow \log_2 \frac{1}{2} = 1$

$\log_2 x = y$

$\log_2 x = y$ this is in the form $f^{-1}(x) = \log_2 x$ or $y = \ldots$

Sketch graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $f^{-1}(x) = \log_2 x$ on the same set of axes.

Both $f$ and $f^{-1}$ are one-to-one functions. The line $y = x$ is the axis of symmetry.

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ACTIVITY 2  (112 marks)

1. Determine the inverse for each of the functions below: (2 marks each)

1.1 \( f(x) = \frac{1}{2}x - 3 \)

1.2 \( g(x) = 5x + 1 \)

1.3 \( h(x) = x^2 + 1 \)

1.4 \( j(x) = -2x^2 + 2 \)

1.5 \( k(x) = 3^x + 1 \)

1.6 \( l(x) = \left(\frac{1}{2}\right)^x - 2 \)

1.7 \( m(x) = 2^{-x} - 2 \)

1.8 \( n(x) = \frac{1}{x+1} - 2 \)

2. For each of the functions in Question 1 above, sketch both the function and the inverse on the same set of axes. Clearly show the asymptotes where necessary. (4 marks each function/4 marks each inverse)

3. State the domain and range for each of the functions and their inverses sketched in Question 2 above. (2 marks each function/ 2 marks each inverse)

<table>
<thead>
<tr>
<th>Function</th>
<th>Inverse</th>
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TOTAL: 112
QUESTION 1

Consider the functions \( f(x) = 2 \times 3^x - 2 \), \( g(x) = -2x + 1 \) and \( h(x) = -(x + 2)^2 + 5 \).

1.1 Solve for \( x \) if \( f(x) = 0 \). (3)

1.2 For which values of \( x \) and \( y \) is \( g(x) = h(x) \)? (5)

1.3 Draw \( f \), \( g \) and \( h \) on the same set of axes, clearly indicating the intercepts with the axis, turning point(s), asymptotes and symmetry lines. (9)

[17]

QUESTION 2

Given: \( f(x) = \log_a x \)

2.1 Determine the value of \( a \), if the point \((27; 3)\) lies on \( f \). (2)

2.2 Determine the equation of \( f^{-1} \) in the form \( y = \ldots \) (2)

2.3 Draw a neat sketch of \( f^{-1} \), showing all intercepts with the axes. Indicate at least one other point on your graph. (2)

2.4 Write down the range of \( h \) if: \( h(x) = f^{-1}(x) + 1 \). (1)

[7]

QUESTION 3

Given: \( h(x) = \frac{12}{x-4} + 6 \) for \( x > 0 \)

3.1 Draw a neat sketch graph of \( h \). Show all intercepts with the axes and asymptotes. (4)

3.2 Write down the equation of \( k \) if \( k \) is the reflection of \( h \) about the \( x \)-axis. (3)

[7]
QUESTION 4

4.1 Sketched below are the functions: \( f(x) = 2x^2 - 6x - 20 \) and \( g(x) = -2x + k \).

Determine:

4.1.1 the coordinates of turning point D. \( (2) \)
4.1.2 the coordinates of A and B. \( (3) \)
4.1.3 the value of \( k \). \( (2) \)
4.1.4 the values of \( p \) if \( 2x^2 - 6x + p = 0 \) has no real roots. \( (2) \)
4.1.5 for which values of \( x \) is \( f(x) \cdot g(x) \leq 0 \). \( (2) \)
4.1.6 the value of \( t \) if \( y = -2x + t \) is a tangent to \( f \). \( (4) \)

4.2 Consider the following two functions: \( p(x) = x^2 + 1 \) and \( r(x) = x^2 + 2x \).

4.2.1 How will you shift \( p \) to become the function \( r \)? \( (3) \)
4.2.2 Write down the range of \( p \). \( (1) \)

TOTAL: 50
ACTIVITY 4 (50 marks)

QUESTION 1

The graphs of \( f(x) = \frac{a}{x + p} + q \), \( g(x) = b \left( \frac{1}{2} \right)^x \) and \( k(x) = mx + c \) are drawn below.

The asymptotes of the hyperbola intersect at \( L(-1 ; 8) \). \( R(-2 ; 12) \) is a common point of \( f \) and \( g \). \( k \) is the line of symmetry of \( f \).

1.1 Determine the values of \( a, p, q, b, m \) and \( c \).

1.2 Determine the equation of the inverse of \( g(x) \) in the form \( y = \ldots \).

1.3 Draw the graph of the inverse of \( \frac{g(x)}{3} \). Clearly indicate the coordinates of the \( x \)-intercept.
QUESTION 2

The graphs of \( f(x) = \frac{4}{x - 3} + 5 \) and \( g(x) = \left(\frac{1}{2}\right)^x + \frac{5}{2} \) are drawn below.

\( P(1 ; 3) \) is a common point of the two graphs. \( O \) is the origin.
The asymptotes of the two graphs are indicated by dotted lines. The two asymptotes intersect at \( K(x ; y) \). \( S \) is the \( y \)-intercept of \( f \) and \( T \) is the \( y \)-intercept of \( g \).

Use the information provided to answer the questions:

2.1 Determine the length of:
   2.1.1 RK (1)
   2.1.2 KM (1)

2.2 For which values or \( x \) is \( g(x) \geq f(x) \) where \( x > 0 \)? (2)

2.3 For which values of \( x \) is \( f(x) > g(x) \) where \( x > 0 \)? (3)

2.4 Determine the length of \( HF \) if \( OH = 5 \) units long. (4)

2.5 Explain why \( g \) is a decreasing function. (2)

2.6 Determine the equation of the axis of symmetry of the hyperbola. (2)

2.7 Write down the domain of \( f \). (2)

2.8 Determine the solution of the following equation where \( x > 0 \):
   \[ 4^{-1}(2)^{-x} - 0.625 = (x - 3)^{-1} \]
   NB: show ALL your working detail and give a reason for your final answer. (5)
QUESTION 3

3.1 Let \( f(x) = -3x^2 \).

3.1.1 How must the domain of \( f \) be restricted such that the inverse of \( f \) is a function again? (2)

3.1.2 Draw a sketch graph of the inverse of \( f \). (2)

3.1.3 Why is the inverse of \( f \) not a function? (1)

3.2 The graphs of \( f^{-1}(x) = \pm \sqrt[2]{a \over x} \) and \( g^{-1}(x) = \log_b x \) are drawn below.

R(25 ; 2) is a common point of the two graphs.

3.2.1 Determine the values of \( a \) and \( b \). (5)

3.2.2 For which values of \( x \) is \( g^{-1} \leq 2 \)? (2)

3.2.3 Determine the equation of the function \( p \) if the graph of \( p \) is obtained by shifting the graph of \( f \) two units to the left. (2)

TOTAL: 50
ACTIVITY 5 (51 marks)

QUESTION 1
Below are the graphs of \( f(x) = -(x + 2)^2 + 6 \) and a straight line \( g \).

- A and B are the \( x \)-intercepts of \( f \) and E is the turning point of \( f \).
- C is the \( y \)-intercept of both \( f \) and \( g \).
- The \( x \)-intercept of \( g \) is D. DE is parallel to the \( y \)-axis.

1.1 Write down the coordinates of E. (2)
1.2 Calculate the coordinates of A. (3)
1.3 M is the reflection of C in the axis of symmetry of \( f \). Write down the coordinates of M. (3)
1.4 Determine the equations of \( g \) in the form \( y = mx + c \). (3)
1.5 Write down the equation of \( g^{-1} \) in the form \( y = \ldots \). (3)
1.6 For which values of \( x \) will \( x(f(x)) \leq 0? \) (4)
QUESTION 2

In the diagram below, the graph of \( f(x) = ax^2 \) is drawn in the interval \( x \leq 0 \). The graph of \( f^{-1} \) is also drawn. \( P(-7; -14) \) is a point on \( f \) and \( R \) is a point \( f^{-1} \).

2.1 Is \( f^{-1} \) a function? Motivate your answer. 

2.2 If \( R \) is the reflection of \( P \) in the line \( y = x \), write down the coordinates of \( R \). 

2.3 Calculate the value of \( a \). 

2.4 Write down the equation of \( f^{-1} \) in the form \( y = \ldots \) 

\[ 8 \]
QUESTION 3
In the diagram below, the graphs of \( f(x) = m^x + k \) and \( g(x) = \frac{-4}{x+2} - 1 \) are drawn.
The two graphs intersect at \( A(0; -3) \). The point \( C(2; 5) \) lies on \( f \) and \( B(-6; 0) \) is the \( x \) - intercept of \( g \).

3.1 Determine:

3.1.1 the equation of \( f \). (4)

3.1.2 the equation of \( h \), the axis of symmetry of \( g \) with a negative gradient. (3)

3.2 Describe the transformation that \( g \) has to undergo to form the graph of

\[ p(x) = \frac{-4}{x+4} + 4 \]  

(2)

3.3 For which value of \( x \) is: \( h(x) \leq g(x) \)? (5)
QUESTION 4

Sketched below are the graphs of $h(x) = \left(\frac{1}{2}\right)^x + q$ and $f(x) = \log_2 x$.

Graph $f$ and the asymptote of $h$ intersect at $B(A; p)$.

4.1 Write down the coordinates of $A$, the $x-intercept$ of $f$.  

4.2 Determine the domain of $f$.  

4.3 Determine the equation of $f^{-1}$ in the form $y = ..$  

4.4 Sketch the graph of $f^{-1}$. Clearly label the intercept(s) with the axes as well as the coordinates of any one other point on the graph.  

4.5 Determine the equation of the asymptote of $h$.  

4.6 Describe, in words, the transformation of $h$ to $f^{-1}$.  

[11]

Total: 51